Symmetry Groups of the Platonic Solids

GEORGE SIVULKA

1 June 2018

Math 109 - Applied Group Theory – Writing in the Major Paper

1 Introduction

1.1 Importance

From the era of Ancient Greece to the modern day, five important 3-dimensional solids have captured the human imagination, playing an important role across various academic pursuits. These five polyhedra, the Platonic Solids, have undoubtedly held a fundamental position in mathematical inquiry. From Plato's first postulation of their existence in his dialogue *The Timaeus*, to Euclid's exploration of their properties in his final book of *The Elements*–they've historically been objects of Mathematical interest.

Still, while geometers have studied their mathematical beauty and unique symmetries for millennia, their influence isn't only limited to Mathematics. They've also played an important role in other fields. For example, in early Cosmology, Johannes Kepler used them to explore his first model of the solar system, a step towards geometric classification of planetary movements that lead to his discovery of the properties of elliptic orbits. Additionally, both Biology and Chemistry make use of their properties and symmetries, through the study of virus morphologies and the structures of the interactions of symmetric molecules respectively. Their implications don't end here; the Platonic Solids are undoubtedly important to study.

This paper serves to offer a mathematical overview of the classification of the symmetries of the Platonic Solids, determining the symmetry groups of each polyhedron explicitly.

1.2 Foundational Background

First, it is important to precisely define the overarching mathematical concepts central to this pursuit.

Definition 1.1 (Platonic Solids)**.** A **Platonic Solid** is a regular, convex polyhedron, with congruent faces, edges, and angles.

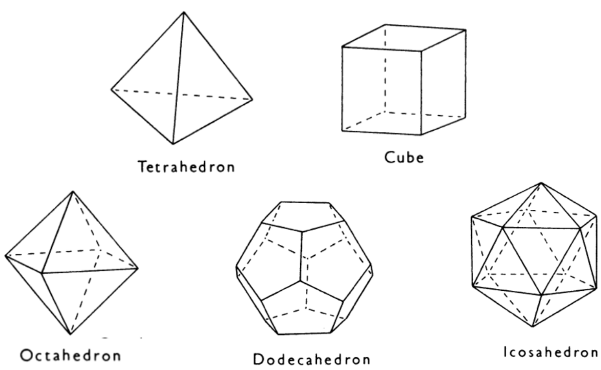

Figure 1: The five Platonic Solids.

Interestingly, there are only five solids that satisfy these conditions: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron (Figure 1). Euclid rigorously proved that these five polyhedra were the only Platonic Solids in the thirteenth book of his *The Elements*, concluding this foundational treatise on Geometry.

Additionally, there are two types of symmetry groups of polyhedra that are fundamental to classifying *all* symmetries of each Platonic Solid. Let X denote the solid under analysis.

Definition 1.2 (Rotational Symmetry Groups)**.** The **Rotational Symmetry Group** of a polyhedron is the group of rotational symmetries of the rigid solid, denoted in this paper as $S_R(X)$.

Definition 1.3 (Full Symmetry Groups)**.** On the other hand, the **Full Symmetry Group** of a polyhedron, is the group of rotational and reflectional symmetries of the solid, denoted in this paper as $S(X)$.

1.3 Overview

Three propositions will be postulated and proved, directly relating to the effort of the classification of these rotational and full symmetry groups for each Platonic Solid. They comprise the theorem proved by this classification:

Theorem 1 (Symmetries of the Platonic Solids)**.** *The symmetries of the Platonic Solids are classified by the following three statements:*

1. *Tetrahedra have a rotational symmetry group isomorphic to* A_4 *and a total symmetry group isomorphic to* S_4
2. *Cubes and Octahedra have a rotational symmetry group isomorphic to* S_4 *and a total symmetry group isomorphic to* $S_4 \times \mathbb{Z}_2$
3. *Dodecahedra and Icosahedra have a rotational symmetry group isomorphic to* A_5 *and a total symmetry group isomorphic to* $A_5 \times \mathbb{Z}_2$

2 Simplifying Concepts

2.1 Relevant Properties of Polyhedra

Additional definitions, theorems, and lemmas offer important background relevant to the classification by helping simplify the task of analyzing these symmetries.

Definition 2.1 (Symmetry Axes)**.** The **symmetry axes** of a polyhedron are lines about which the solid can be rotated by some angle such that the polyhedron's new orientation is seemingly identical to its starting position.

Definition 2.2 (Symmetry Planes)**.** Similarly, **symmetry planes** of a polyhedron are external two dimensional surfaces upon which reflection of the polyhedra returns a new orientation seemingly identical to its starting position.

Definition 2.3 (Dual Polyhedra)**.** The **dual polyhedron** of a regular solid is another polyhedron such that the faces and vertices of the two occupy complementary locations. This can be constructed by connecting the centers of each face of the solid, inscribing this new dual polyhedron within the original solid.

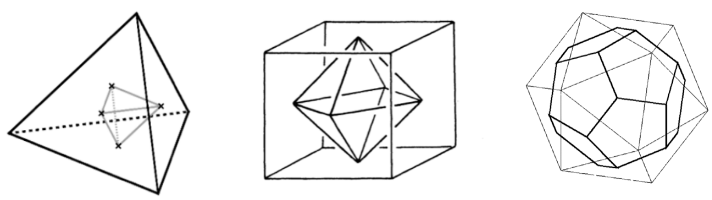

Figure 2: The dual polyhedra pairs of all the Platonic Solids.

Note that the dual of each Platonic Solid is also a platonic Solid. Tetrahedra are self-dual, or dual to themselves, cubes and octohedra are dual, and dodecahedra and icosahedra are also dual (Figure 2).

2.2 Relevant Lemma and Theorems

Using the above concepts, we can relate the symmetry groups of Platonic Solids that are dual:

Lemma 1. *If two polyhedra are dual, then they share the same symmetry groups.*

Proof. This follows directly from the observation that dual polyhedra share the same axes and planes of symmetry. □

Now, as dual solids–such as cubes and octohedra or dodecahedra and icosahedra–share the same symmetry groups, all symmetry groups of the Platonic Solids can be determined once the symmetry groups of tetrahedra, cubes, and dodecahedra are known.

Additionally, we can relate the full symmetry groups of cubes and dodecahedra with their rotational symmetry groups, using the concept of a direct product and results from introductory Group Theory.

First note **Theorem 10.2.** in Armstrong's *Groups and Symmetry*, proved on page 54: *If H and K are subgroups of G for which $HK = G$, if they have only the identity element in common, and if every element of H commutes with every element of K, then G is isomorphic to $H \times K$.*

Relating these concepts and this theorem to the Full Symmetry Group of a solid, we have the result:

Theorem 2. *The full symmetry group, S(X), for some solid, X, is equal to the direct product of the rotational symmetry group $S_R(X)$ and $\mathbb{Z}_2$.*

Proof. Let $f_j : \mathbb{R}^3 \to \mathbb{R}^3$ be a central inversion, or function sending a vector $\mathbf{x}$ to $\mathbf{-x}$. Note that, aside from the tetrahedron, any regular solid centered around the origin has a reflectional symmetry equal to this central inversion. Let G be the full symmetry group of X and H be the rotational symmetry group of X. We know that $\langle f_j \rangle$, the group generated by the central inversion, is also a subgroup of G by the definition of G. Moreover, it shares only the identity element with H. We can see that every element of G commutes with every element of H by first noting that elements of H can be expressed a subgroup of the Orthonormal Matrix Group in three dimensions because these matrices explicitly represent non-scaling rotations in three-space. Likewise, elements of G can be expressed as a scalar multiple on that matrix of 1 or -1. Scalar-matrix multiplication is commutative on these elements equivalent to rotations and reflections, so elements of G and H must also commute. Thus, by Theorem 10.2 stated above, G is isomorphic to $H \times \langle f_j \rangle$. Now, all we need to show is that $\langle f_j \rangle$ is isomorphic to $\mathbb{Z}_2$. Let y be some element in $\langle f_j \rangle$ and $\phi : \langle f_j \rangle \to \mathbb{Z}_2$ be an isomorphism defined as:

$$\phi(y) = \begin{cases} 0, y = \mathbf{x} \\ 1, y = \mathbf{-x} \end{cases}$$

Clearly, ϕ is a bijection and it also satisfies the homomorphism criterion for all pairs of elements: $\phi(0 + 0) = x = \phi(0)\phi(0)$, $\phi(0 + 1) = -x = \phi(0)\phi(1)$, and $\phi(1 + 1) = x = \phi(1)\phi(1)$. As such, there exists an isomorphism between $\langle f_j \rangle$ and $\mathbb{Z}_2$. Therefore, $S(X)$ is isomorphic to $S_R(X) \times \mathbb{Z}_2$, and the theorem holds. □

By **Lemma 1** and **Theorem 2** above, our task has been reduced to only finding the rotational and full symmetry groups of the tetrahedron and the rotational symmetry groups of the cube and dodecahedron.

2.3 The Tetrahedron

Proposition 1. *Tetrahedra have a rotational symmetry group isomorphic to A_4 and a total symmetry group isomorphic to S_4*

First, note that a tetrahedron has four vertices. For each permutation of these vertices, there exists a symmetry in the total symmetry group. Specifically, the first vertex can take four different positions. The second vertex can then end up in any of the three remaining positions via rotation. The third vertex must then take any of the final two positions by reflection, and now the position of the fourth vertex remains fixed. Therefore, under rotations and reflections, the Tetrahedron has 4 * 3 * 2 * 1 or 24 total symmetries. Observe that the order of S_4, the Permutation Group of order 4, also has order 24.

Now, each vertex can be labeled from 1 to 4, and thus, permutations of vertex positions can be expressed under cyclic notation. Using this notation, first the rotational symmetries can be listed out. A tetrahedron has two axes of symmetry, one passing through the center of one face and the vertex right above it, and another passing through the center of one edge and the perpendicular edge adjacent to it (Figure 3). We can label these axes of symmetry as L and M, respectively.

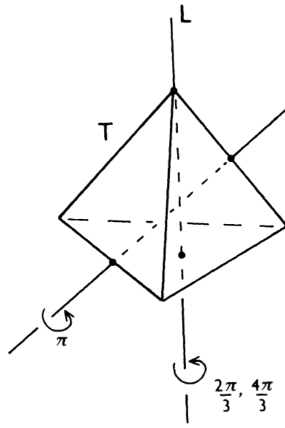

Figure 3: The axes of symmetry of the Tetrahedron.

Clearly, an axis of type L permutes only three vertices, and thus all three cycles of the vertex elements 1, 2, 3, and 4 describe rotations along such axes. Thus, the 8 possible three cycles (123), (132), (124), (142), (134), (143), (234), and (243) correspond to the possible 120 degree symmetry rotations. On the other hand, an axis of type M permutes all four vertices, swapping them in pairs. Thus, the three possible products of two disjoint transpositions, (12)(34), (13)(24), and (14)(23) correspond to elements of the rotational symmetries wherein the solid is revolving 180 degrees. The final rotational symmetry, the identity–not rotating the shape at all–corresponds to the cyclic notation describing no permutations, (). Note that these 12 possible rotational symmetries directly correspond to all even order elements of S_4, otherwise known as the Alternating Group A_4. Clearly when two rotations r and r' prompt permutations p and p' respectively, their composed rotation rr' prompts the permutation pp', exhibiting a homomorphism. Moreover, the injective and surjective mapping explicitly listed out above exhibits a bijection. Thus, via this correspondence, the group of rotational symmetries of the Tetrahedron are isomorphic to A_4.

On a similar note, the possible reflections of the tetrahedron can also be expressed using cyclic notation. Note that the only possible tetrahedral plane of symmetry would intersect both the midpoint of an edge and the opposite vertices of the two faces containing that edge. Equivalently, a plane of symmetry must be spanned by any two L and M axes of symmetry, and would swap any two vertices of the tetrahedron not contained in this plane. Thus all six transpositions of S_4, (12), (13), (14), (23), (24), and (34), correspond to a reflectional symmetry. Now, the only elements that don't correspond to a single reflection or rotation are remaining four cycles (1234), (1243), (1324), (1342), (1423), and (1432). We can see that (1234) is equivalent to the product (123)(34), and moreover, that the corresponding movement matches up with the composition of a reflection and rotation on the solid. Similarly all other elements of S_4 can be mapped to by elements generated by rotations and reflections, and this map is thus surjective. Now, as all 24 elements of S_4 map to the 24 possible rotation and reflection symmetries of the Tetrahedron, and compositions of these elements directly correspond on both sides of the mapping, the full group of symmetries is isomorphic to S_4.

2.4 The Cube and Octahedron

Proposition 2. *Cubes and Octahedra have a rotational symmetry group isomorphic to* S_4 *and a total symmetry group isomorphic to* $S_4 \times \mathbb{Z}_2$

Under rotational symmetries opposite vertices in a cube can be paired together, as for any rigid rotation of a vertex in a cube, its opposite vertex must move accordingly to remain opposite. Thus, in similar fashion to the argument for the number of *total* symmetries of a tetrahedron, we can claim that the number of *rotational* symmetries of a cube is the number ways you can permute these 4 pairs of vertices–if a rotation permuting all vertex couples can be found. Below, we will show that rotations do permute every vertex; it follows that the number of rotational symmetries is 24.

There exist three types of axes of symmetry on the cube (Figure 4). The first type, denoted here as L, intersects the midpoint of two faces of the cube. There are three such axes, and each allows three rotational symmetries, by 90, 180, and 270 degree rotations respectively. Thus there exist nine rotations about L axes. Another axis type intersects the midpoint of two opposite edges, denoted here as M. There are six such axes, and each has one rotational symmetry of 180 degrees, so there are 6 rotations about M axes. Finally, the last axis type, denoted here as N, intersects two opposite vertices, and there are 4 opposite vertex pairs as previously stated. On each N axis, there are two allowed symmetries created by rotating the solid by 120 and 240 degrees. Thus, N axes have 8 allowed rotations. In sum, all possible rotations–9, 6, and 8 for each axis type–plus the identity add up to 24 symmetry elements.

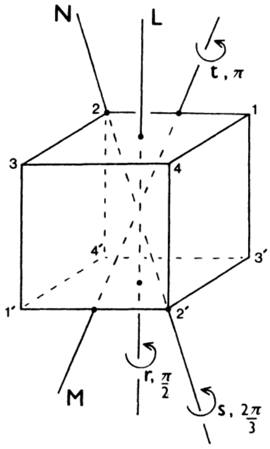

Figure 4: The axes of symmetry of the cube.

Using this information, it is possible to show that group of rotations above is isomorphic to S_4. Numbering the corners on only one face of the cube from 1 to 4 allows us to then number their corresponding opposite vertices from 1' to 4' respectively, differentiating the four permutable constituents of the cube. Now, permutations of these elements directly correspond to permutations in S_4. Note how rotation about axis types L, M, and N return four cycles, transpositions, and three cycles respectively. Additionally, observe how a product of two rotations clearly induces the correct product of two permutations in S_4 by analysis. A surjective correspondence is bijective if it maps two sets of equal size and the number of rotations found above, 24, is exactly equal to the number of elements in S_4. Thus the group of rotations of the cube is isomorphic to S_4.

Applying **Lemma 1**, as the rotational symmetry group of the cube is isomorphic to S_4, the rotational symmetry group of its dual, the octahedron, is also isomorphic to S_4.

Now, by **Theorem 2**, the full symmetry groups of a cube and a octahedron must be isomorphic to $S_4 \times \mathbb{Z}_2$.

2.5 The Dodecahedron and Icosahedron

Proposition 3. *Dodecahedra and icosahedra have a rotational symmetry group isomorphic to* A_5 *and a total symmetry group isomorphic to* $A_5 \times \mathbb{Z}_2$

First, in order to determine the number of rotational symmetries of the dodecahedron, we can count the number of ways we can permute its vertices. Note that the solid has 20 vertices, and each vertex is adjacent to 3 other vertices. Thus, there are 20 places to map our first vertex to. Taking a second vertex that was adjacent to the first vertex, there are only new 3 adjacent spots it can map to. Once two adjacent verities are fixed, all other vertices under a rigid transformation are then determined; the number of possible rotations of the dodecahedron is 20 * 3 or 60. Observe that this is equal to the order of A_5.

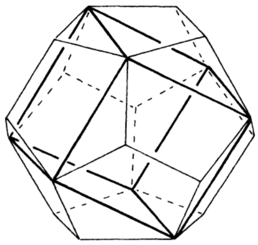

Figure 5: Inscribed cubes of the dodecahedron.

Similar to the way rotations permute opposite *pairs* of vertices of the cube as noted above, rotations of the dodecahedron permute five inscribed *cubes* amongst each other. Observe that the edges of each cube are diagonals of every pentagonal face of the dodecahedron (Figure 5). Moreover, each of the five possible diagonals on every pentagonal face corresponds to one of the five inscribed permutable cubes. We can number each cube by numbering these diagonals on the topmost face of the dodecahedron, starting at the nearest diagonal, and labeling them from 1 to 5 in a clockwise fashion–so that Figure 5 portrays the 5th diagonal and thus cube 5.

Now, note that the cube has axes of symmetry that intersect opposite vertices in pairs; there are 10 such axes for the 20 vertices. Moreover, as each vertex connects three edges, and they must map to each other in a rotational symmetry about that vertex, these axes have only 2 rotational symmetry elements of 120 and 240 degrees. Therefore there are 20 total rotational symmetries among these axes, and we can show they correspond to the 20 3-cycles in A_5.

Choosing one such axes of symmetry, we can see that its rotations fix the two inscribed cubes whose N axes (as labeled in the last section, Figure 4) intersect the same two vertices. Note that the N axis has rotational symmetries of 120 and 240 degrees, equivalent to the rotations exhibited by the dodecahedron we investigate. Now, there are three remaining inscribed cubes not-fixed by rotations on each axes of symmetry and thus must be sent to each other. These cubes can be represented by their numbered face diagonal per the labeling scheme above. Thus each rotation among these axes directly corresponds to a permutation of three cubes, or a 3-cycle in S_5. In fact, as there exist 20 unique rotational symmetries along the 10 diagonals, 20 unique 3-cycles can be expressed. There are a total of 20 unique 3-cycles possible in S_5, so these rotational elements must correspond to *all* 3-cycles in S_5.

Finally, by **Theorem 6.5** on page of Armstrong, *for $n \geq 3$ the 3-cycles generate A_n*, so all the 3-cycle permutations mapped to in S_5 generate A_5. This is clearly a homomorphism, as combinations of rotations clearly correspond to associated permutation groups in A_5 by definition. Moreover, the map criteria described above details a bijection; it is surjective as all elements in A_5 could be mapped to by **Theorem 6.5**, and both sets have order 60 as shown at the start of this section. Thus there exists an isomorphism between A_5 and the rotational symmetry group of the dodecahedron.

Applying **Lemma 1**, as the rotational symmetry group of the dodecahedron is isomorphic to A_5, the rotational symmetry group of its dual, the icosahedron, is also isomorphic to A_5.

Now, by **Theorem 2**, the Full Symmetry Groups of a dodecahedron and a icosahedron must be isomorphic to $A_5 \times \mathbb{Z}_2$.

3 Conclusion

The task of classifying the symmetry groups of the Platonic solids was greatly reduced by the concepts of dual polyhedra and central inversion. Tetrahedra were found to have a rotational symmetry group isomorphic to A_4 and a total symmetry group isomorphic to S_4. Cubes and octahedra have a rotational symmetry group isomorphic to S_4 and a total symmetry group isomorphic to $S_4 \times \mathbb{Z}_2$. Finally, dodecahedra and icosahedra have a rotational symmetry group isomorphic to A_5 and a total symmetry group isomorphic to $A_5 \times \mathbb{Z}_2$, completing the classification of these symmetries.

References

[1] M. A. Armstrong, Groups and Symmetry, Springer, New York, 1988

Note: all unedited or non-original figures sourced form Armstrong.

[2] O'Connor, John. Symmetry Groups of Platonic Solids. 2003, www-history.mcs.st-and.ac.uk/ john/geometry/Lectures/L10.html.